Oracle complexity classes and local measurements on physical Hamiltonians

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(Graduated — Looking for graduate positions)

Two themes

Two themes from quantum Hamiltonian complexity:

k-LH	Physical problems motivating complexity classes		APX-SIM
$\Updownarrow$	$\Downarrow$	$\Uparrow$	$\Updownarrow$
QMA	Complexity classes prompting physical results		$\mathsf{P}^{\mathsf{QMA[log]}}$

Physical problems motivating complexity classes

Estimating local measurements

Definition (**APX-SIM**$(H, A, k, l, a, b, \delta)$ [Ambainis, 2014])

Given:

- k-local Hamiltonian H on n qubits
- l-local observable A
- $a, b, \delta \in \mathbb{R}$ such that $b - a \geq \frac{1}{\text{poly}(n)}$ and $\delta \geq \frac{1}{\text{poly}(n)}$,

Decide:

- If H has a ground state $|\psi\rangle$ satisfying $\langle\psi| A |\psi\rangle \leq a$, output YES.
- If for all $|\psi\rangle$ satisfying $\langle\psi| H |\psi\rangle \leq \lambda(H) + \delta$, it holds that $\langle\psi| A |\psi\rangle \geq b$, output NO.

- [Ambainis, 14] showed APX-SIM is $\text{P}^{\text{QMA[log]}}$-complete.
- [Gharibian and Y., 2016] showed completeness holds for single-qubit measurements!

Oracle complexity

Definition ([Ambainis, 2014])

$\mathbf{P}^{\mathbf{QMA[log]}}$ is the class of decision problems decidable by a P machine with the ability to query a QMA oracle up to $O(\log n)$ times.

Intuitively, $\mathrm{P}^{\mathrm{QMA[log]}}$ is "slightly harder" than QMA.

Formally, $\mathrm{QMA} \subseteq \mathrm{P}^{\mathrm{QMA[log]}} \subseteq \mathrm{PP}$ [Gharibian and Y., 2016].

($\mathrm{QMA} \subseteq \mathrm{PP}$ previously known [Kitaev and Watrous, 2000])

$\mathrm{P}^{\mathrm{QMA[log]}}$ is interesting — *surprisingly*, in contrast to $\mathrm{P}^{\mathrm{NP[log]}}$ — because it characterizes physically interesting problems: APX-SIM, APX-2-CORR, SPECTRAL-GAP, ...

First result

Theorem 1

$\mathsf{P}^{\mathsf{QMA[log]}} = \mathsf{P}^{||\mathsf{QMA}}$.

Definition

$\mathbf{P}^{||\mathbf{QMA}}$ is $\mathsf{P}^{\mathsf{QMA[log]}}$ but with up to polynomially many parallel / non-adaptive queries.

- Analogous classical result: $\mathsf{P}^{\mathsf{NP[log]}} = \mathsf{P}^{||\mathsf{NP}}$ [Beigel, 1991].
- $\mathsf{P}^{\mathsf{QMA[log]}} \subseteq \mathsf{P}^{||\mathsf{QMA}}$: Proof exactly the same as for $\mathsf{P}^{\mathsf{NP[log]}} \subseteq \mathsf{P}^{||\mathsf{NP}}$.
  We will focus on the reverse containment . . .

Novel proof technique $\mathsf{P}^{\mathsf{QMA[log]}} \supseteq \mathsf{P}^{||\mathsf{QMA}}$

Classically, show $\mathsf{P}^{\mathsf{NP[log]}} \supseteq \mathsf{P}^{||\mathsf{NP}}$ directly, using NP oracle to count number of parallel queries which are YES-instances.

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This technique fails in the quantum case!

The P machine may make invalid queries, i.e which violate the promise of k-LH, with $\lambda_{\mathsf{min}} \in (a, b)$.

The oracle is unpredictable given invalid queries.

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New technique:

Use $\mathrm{APX\text{-}SIM} \in \mathsf{P}^{\mathsf{QMA[log]}}$, and prove $\mathrm{APX\text{-}SIM}$ is $\mathsf{P}^{||\mathsf{QMA}}$-hard, to show $\mathsf{P}^{\mathsf{QMA[log]}} \supseteq \mathsf{P}^{||\mathsf{QMA}}$.

How? We adapt the improved "query Hamiltonian" construction of [Gharibian and Y., 2016].

Bonus: Simplifies $\mathsf{P}^{\mathsf{QMA[log]}}$-completeness proof

Showing $\mathrm{APX\text{-}SIM} \in \mathsf{P}^{\mathsf{QMA[log]}}$ is relatively simple [A14].

[Ambainis, 2014] showed $\mathrm{APX\text{-}SIM}$ is $\mathsf{P}^{\mathsf{QMA[log]}}$-hard using their "query Hamiltonian" construction:

$$H_{\text{query}} = \sum_{i=1}^{m} \frac{1}{4^{i-1}} \sum_{y_1,\ldots,y_{i-1}} \bigotimes_{j=1}^{i-1} |y_j\rangle\langle y_j|_{\mathcal{X}_j} \otimes \left(2\epsilon\,|0\rangle\langle 0|_{\mathcal{X}_i} \otimes I_{\mathcal{Y}_i} + |1\rangle\langle 1|_{\mathcal{X}_i} \otimes H_{\mathcal{Y}_i}^{i,y_1\cdots y_{i-1}}\right)$$

We show $\mathrm{APX\text{-}SIM}$ is $\mathsf{P}^{||\mathsf{QMA}}$-hard, using the simplified Hamiltonian:

$$H'_{\text{query}} = \sum_{i=1}^{m} \left(2\epsilon\,|0\rangle\langle 0|_{\mathcal{X}_i} \otimes I_{\mathcal{Y}_i} + |1\rangle\langle 1|_{\mathcal{X}_i} \otimes H^{i}_{\mathcal{Y}_i}\right)$$

~~Physical problems motivating complexity classes~~

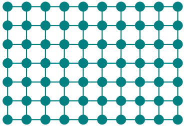
$\Downarrow$

Complexity classes prompting physical results

Physically 'nice' Hamiltonians

k-LH: originally shown QMA-complete given 5-local H as input [Kitaev, 1999].

- Still holds for H restricted to interactions on a 2D square lattice, [Oliveira and Terhal, 2008],
- and H restricted to a 1D line, [Aharonov, Gottesman, et al., 2009],
- and H restricted to the Heisenberg model [Cubitt, Montanaro, and Piddock, 2017].



We ask whether similar results hold for APX-SIM and $P^{QMA[\log]}$.

Second result

Theorem 2

APX-SIM is $\mathsf{P}^{\mathsf{QMA[log]}}$-complete for $O(1)$-local observables and Hamiltonians able to efficiently simulate spatially sparse Hamiltonians.

Definition (Informal — [Cubitt, Montanaro, and Piddock, 2017])

H' is a **simulation** of Hamiltonian H if there exists a local isometry $V = \bigotimes_i V_i$ that maps eigenvectors and eigenvalues of H to eigenvectors and eigenvalues of H' with "sufficiently small errors".

Ex. [Cubitt, Montanaro, and Piddock, 2017] show that Hamiltonians in the XY model can efficiently simulate spatially sparse Hamiltonians.

Our strategy

Definition (**APX-SIM** [Ambainis, 2014])

- If H has a ground state $|\psi\rangle$ satisfying $\langle\psi| A |\psi\rangle \leq a$, output YES.
- If for all $|\psi\rangle$ satisfying $\langle\psi| H |\psi\rangle \leq \lambda(H) + \delta$, it holds that $\langle\psi| A |\psi\rangle \geq b$, output NO.

Definition (**APX-SIM2**)

If for all $|\psi\rangle$ satisfying $\langle\psi| H |\psi\rangle \leq \lambda(H) + \delta$,

- $\langle\psi| A |\psi\rangle \leq a$, output YES.
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1. *Show $\mathrm{APX\text{-}SIM2}$ is $\mathrm{P}^{||\mathrm{QMA}}$-complete for spatially sparse Hamiltonians.*

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1. *Show $\mathrm{APX\text{-}SIM2}$ is $\mathrm{P}^{||\mathrm{QMA}}$-complete for spatially sparse Hamiltonians.*
2. Show if H' simulates H, then $\mathrm{APX\text{-}SIM2}(H)$ reduces to $\mathrm{APX\text{-}SIM2}(H')$.

Proof sketch of step 1

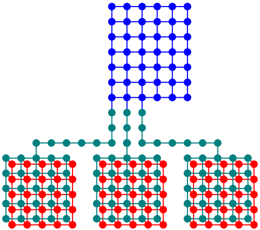

Figure: H'_{Kitaev}, H''_{query}, H_{stab}

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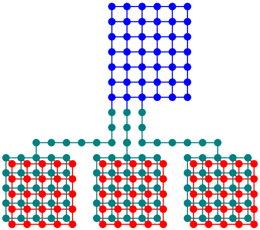

Figure: H'_{Kitaev}, H''_{query}, H_{stab}

Goal: Show $\mathrm{APX\text{-}SIM2}$ is $\mathsf{P}^{||\mathsf{QMA}}$-complete for spatially sparse Hamiltonians.

Modify construction $H = H_{\mathsf{Kitaev}} + H'_{query}$ from Theorem 1 by:

- Reduce H_{Kitaev} using spatially sparse QMA-hardness construction of [Oliveira and Terhal, 2008].
- Reduce the oracle queries, which H'_{query} acts on, to spatially sparse similarly.
- Design a stabilizer term, H_{stab}, to join the two.

Summary

Theorem 1

$\mathsf{P}^{\mathsf{QMA[log]}} = \mathsf{P}^{||\mathsf{QMA}}$.

Theorem 2

APX-SIM is $\mathsf{P}^{\mathsf{QMA[log]}}$-complete for $O(1)$-local observables and Hamiltonians able to efficiently simulate spatially sparse Hamiltonians.

Takeaway: Estimating $O(1)$-qubit measurements against ground states of systems as simple as the Heisenberg model is harder than QMA.

Open questions:

- What if we switch the base class P with other classes (classical or quantum)?
- What if we switch the QMA oracle with other quantum oracles?
- Identify additional $\mathsf{P}^{\mathsf{QMA[log]}}$-complete problems and natural inputs.