Oracle complexity classes and local measurements on physical Hamiltonians

S. Gharibian*,‡ S. Piddock† J. Yirka‡

* University of Paderborn, Germany. sevag.gharibian@upb.de † University of Bristol, UK. stephen.piddock@bristol.ac.uk ‡ Virginia Commonwealth University, USA (graduated). yirkajk@vcu.edu

Introduction

In 2014, Ambainis [1] formalized a very natural physical problem: Given local Hamiltonian *H* and observable *A*, how difficult is it to *simulate* the measurement *A* on the ground space of *H*? Formally:

APX-SIM [1]: Given k-local Hamiltonian H and l-local observable A, and $a, b, \delta \in \mathbb{R}$ such that $b - a \geq \frac{1}{\mathrm{poly}(n)}$, $\delta \geq \frac{1}{\mathrm{poly}(n)}$, for n the number of qubits H acts on, decide whether:

- YES: there exists a ground state $|\psi\rangle$ of H such that $\langle\psi|A|\psi\rangle \leq a$;
- NO: for all $|\psi\rangle$ s.t. $\langle\psi|H|\psi\rangle \leq \lambda_{\min}(H) + \delta$, it holds that $\langle\psi|A|\psi\rangle \geq b$.

Ambainis [1] showed that APX-SIM is $\mathrm{P}^{\mathrm{QMA[log]}}$-complete for $O(\log n)$-local *H* and *A*, where:

$\mathbf{P}^{\mathbf{QMA[log]}}$**:** the set of problems decidable in polynomial time given $O(\log n)$ queries to a QMA oracle.

Improvements:

- G. and Y. showed [2] showed APX-SIM remains $\mathrm{P}^{\mathrm{QMA[log]}}$-complete even for **5-local Hamiltonian** $\boldsymbol{H}$ and **1-local measurement** $\boldsymbol{A}$.
- [2] also showed $\mathrm{P}^{\mathrm{QMA[log]}}$ is only "slightly harder" than QMA, in that $\mathrm{P}^{\mathrm{QMA[log]}} \subseteq \mathrm{PP}$.

Motivating question:
Does simulating measurements on ground spaces (APX-SIM) remain $\mathrm{P}^{\mathrm{QMA[log]}}$-complete for more physically motivated local Hamiltonians?

Main Results

We answer our motivating question positively. This is done via Result 3, which first requires Results 1 and 2.

Result 1: Parallel vs. adaptive queries. We show that $O(\log n)$ *adaptive* queries to a StoqMA or QMA oracle is equivalent to $\mathrm{poly}(n)$ *parallel* queries to the oracle. Formally:

$$\mathrm{P}^{\mathrm{StoqMA[log]}} = \mathrm{P}^{||\mathrm{StoqMA}} \text{ and } \mathrm{P}^{\mathrm{QMA[log]}} = \mathrm{P}^{||\mathrm{QMA}}.$$

Result 2: Complexity of ∀-APX-SIM under simulations. We show that the complexity of a *seemingly* easier problem, **∀-APX-SIM** (see proof techniques), is preserved under "simulations" (in the sense of [3]).

Combined with known simulation results [3], this yields several complexity classifications for our original problem APX-SIM: It is in P, or is $\mathrm{P}^{\mathrm{NP[log]}}$, $\mathrm{P}^{\mathrm{StoqMA[log]}}$, or $\mathrm{P}^{\mathrm{QMA[log]}}$-complete for several Hamiltonian families.

Result 3: Complexity of APX-SIM for physical Hamiltonians.
Leveraging Result 1, we show APX-SIM is $\mathrm{P}^{\mathrm{QMA[log]}}$-complete for spatially-sparse H.
Combining with Result 2, APX-SIM remains $\mathrm{P}^{\mathrm{QMA[log]}}$-complete for any Hamiltonian family which can efficiently simulate spatially-sparse H.

Punchline: APX-SIM is $\mathrm{P}^{\mathrm{QMA[log]}}$-complete on physically motivated models like the Heisenberg anti-ferromagnetic interaction on a 2D lattice.

Proof techniques

Our approach proceeds in two high-level steps:

1. Give $\mathrm{P}^{\mathrm{QMA[log]}}$ an equivalent characterization in terms of *polynomially many parallel* queries, i.e. $\mathrm{P}^{||\mathrm{QMA}}$, which eases the analysis of using Ambainis's [1] query Hamiltonian construction (Result 1).
2. We wish to apply the "simulation" framework of [3] to show that APX-SIM is $\mathrm{P}^{||\mathrm{QMA}}$-complete on physically motivated H. Four substeps:
   a) Introduce intermediary problem, ∀-APX-SIM (see def. below).
   b) Show that simulation preserves the complexity of ∀-APX-SIM (Result 2).
   c) Show that ∀-APX-SIM is $\mathrm{P}^{||\mathrm{QMA}}$-complete for spatially sparse H.
   d) Apply existing simulation results [3] to obtain Result 3, i.e. that APX-SIM is $\mathrm{P}^{\mathrm{QMA[log]}}$-complete for various physically motivated models.

Notes:

- **Simulation** [3]: H_0 is a simulation of Hamiltonian H if there exists an efficiently computable local isometry $V = \bigotimes_i V_i$ that maps eigenvectors and eigenvalues of H to those of H_0 with "sufficiently small errors".
- **∀-APX-SIM**: Defined as APX-SIM but with "∀ low-energy states $|\psi\rangle$" in the YES case. This problem is more robust than APX-SIM to perturbations in the ground space.

$\mathbf{P}^{\mathbf{QMA[log]}} = \mathbf{P}^{||\mathbf{QMA}}$

Classical Theorem [4]**:** $\mathrm{P}^{\mathrm{NP[log]}} = \mathrm{P}^{||\mathrm{NP}}$.
Forward direction $\mathrm{P}^{\mathrm{C[log]}} \subseteq \mathrm{P}^{||\mathrm{C}}$ was shown for all classes C.
Generate all $2^{\log(n)} = \mathrm{poly}(n)$ possible queries and pass to $\mathrm{P}^{||\mathrm{C}}$ machine.
Reverse direction $\mathrm{P}^{\mathrm{NP[log]}} \supseteq \mathrm{P}^{||\mathrm{NP}}$ was shown using NP oracle to binary search the number of parallel queries which are YES-instances.
This technique fails in the quantum case! The P machine may make *invalid* queries, i.e. which violate promise gap, ex. k-LH with $\lambda_{\min} \in (a, b)$.

Theorem: $\mathrm{P}^{\mathrm{QMA[log]}} = \mathrm{P}^{||\mathrm{QMA}}$

Proof: Forward direction as above. To show the reverse direction (⊇), we leverage a *hardness* result. Given that APX-SIM ∈ $\mathrm{P}^{\mathrm{QMA[log]}}$, we prove that APX-SIM is $\mathrm{P}^{||\mathrm{QMA}}$-hard.
We do so by adapting the "query Hamiltonian" constructions of [1], [2]:

$$H'_{\mathrm{queries}} = \sum_{i=1}^{m} \left(2\epsilon|0\rangle\langle 0|_{\mathcal{X}_i} \otimes I + |1\rangle\langle 1|_{\mathcal{X}_i} \otimes H^i_{\mathrm{query}}\right)$$

Additional benefits:

- Here, we use the classical Cook-Levin construction rather than Kitaev's circuit-to-Hamiltonian (as in [2]). This yields *O*(1) promise gap.
- This indirect method can be seen as simplifying the original proof that APX-SIM is $\mathrm{P}^{\mathrm{QMA[log]}}$-complete, by greatly easing analysis of H'_{queries}.

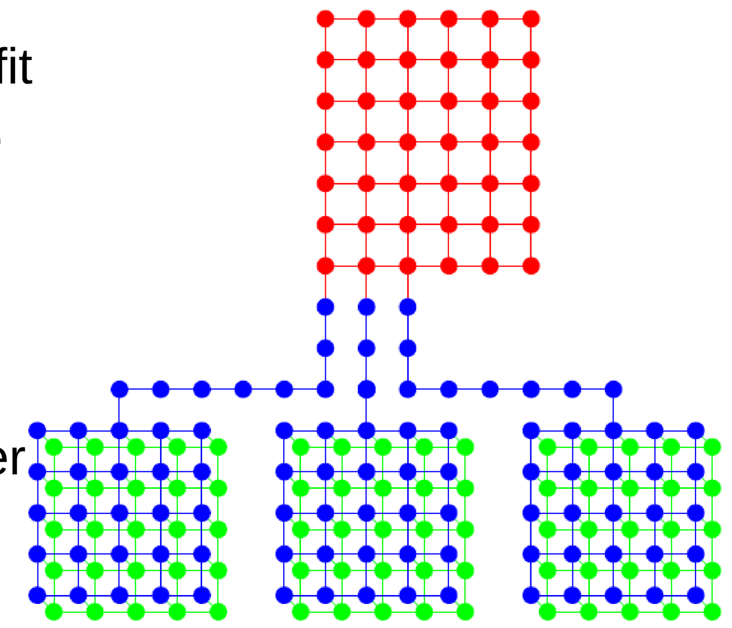
This technique works for any class $\mathbf{C}$ for which there exists a family of Hamiltonians for which $\boldsymbol{k}$-LH is $\mathbf{C}$-complete! ex. NP, StoqMA, or QMA

Complexity of APX-SIM for physical H

Theorem: APX-SIM is $\mathrm{P}^{\mathrm{QMA[log]}}$-complete even for H restricted to a spatially-sparse interaction graph (in the sense of [5]).

Proof: We modify the hardness construction from Result 1 so that it is spatially-sparse. There are 3 Hamiltonian terms,

$$H_{\mathrm{final}} = H_{\mathrm{Cook-Levin}} + H_{\mathrm{queries}} + H_{\mathrm{stab}}:$$



- $H_{\mathrm{Cook-Levin}}$ is already spatially sparse (on a 2D lattice, in fact) – another benefit of our modifications in proving Result 1.
- Since k-LH is QMA-hard for H on a 2D lattice [5], we assume all query instances H^i_{query} in H_{queries} are spatially-sparse.
- But, H_{queries} also has an answer register such that answer qubit $\mathcal{X}_i$ interacts with every qubit in H^i_{query}; this is not spatially-sparse. We "spread out" the answer register to a grid of qubits overlaid on the query register, and introduce a stabilizing term H_{stab}.

Combined with Result 2 and prior results for simulating spatially-sparse Hamiltonians, we obtain many corollaries. Here are a few:
Corollaries: APX-SIM is $\mathrm{P}^{\mathrm{QMA[log]}}$-complete even for H of XY interactions; for H of Heisenberg interactions; or for H on a 2D square lattice.

Discussion

Key takeaways:

1. The natural problem of simulating O(1)-qubit measurements against ground states of physically motivated systems, such as the Heisenberg XY and antiferromagnetic interaction on a 2D lattice, is harder than QMA.
2. Working with *parallel* queries, as opposed to *adaptive* queries, simplifies circuit-to-Hamiltonian constructions.

Open questions:

- What other results for k-LH / QMA can be extended to $\mathrm{P}^{\mathrm{QMA[log]}}$?
- Identify additional $\mathrm{P}^{\mathrm{QMA[log]}}$-complete problems and physical inputs.
- What happens if we switch the P base with other classes (classical or quantum) (ex. $\mathrm{BQP}^{\mathrm{QMA[log]}}$)?
- What if we use other quantum oracles (ex. $\mathrm{P}^{\mathrm{BQP[log]}}$)?

References

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