Introduction

In 2014, Ambainis [1] formalized a very natural physical problem: Given local Hamiltonian $H$ and observable $A$, how difficult is it to simulate the measurement $A$ on the ground space of $H$? Formally:

**APX-SIM [1]:** Given $k$-local Hamiltonian $H$ and $l$-local observable $A$, and $a, b, δ ∈ \mathbb{R}$ such that $b - a ≥ \frac{1}{(poly(n))}$, for $n$ the number of qubits $H$ acts on, decide whether:

- **YES:** there exists a ground state $|\psi\rangle$ of $H$ such that $\langle \psi | A | \psi \rangle ≤ a$;
- **NO:** for all $|\psi\rangle$ s.t. $\langle \psi | H | \psi \rangle ≤ \lambda_{min}(H) + \delta$, it holds that $\langle \psi | A | \psi \rangle ≥ b$.

Ambainis [1] showed that APX-SIM is $P^{\text{QMA}[\text{log}]}$-complete for $O(\log n)$-local $H$ and $A$, where:

\[\text{P}^{\text{QMA}[\text{log}]} = \text{the set of problems decidable in polynomial time given } O(\log n) \text{ queries to a QMA oracle.}\]

**Improvements:**
- G. and Y. showed [2] showed APX-SIM remains $P^{\text{QMA}[\text{log}]}$-complete even for 5-local Hamiltonian $H$ and $l$-local measurement $A$.
- [2] also showed $P^{\text{QMA}[\text{log}]}$ is only “slightly” harder than QMA, in that $P^{\text{QMA}[\text{log}]} ⊆ P^{\text{QMA}}$.

**Motivating question:** Does simulating measurements on ground spaces (APX-SIM) remain $P^{\text{QMA}[\text{log}]}$-complete for more physically motivated local Hamiltonians?

Main Results

We answer our motivating question positively. This is done via Result 3, which first requires Results 1 and 2.

**Result 1:** Parallel vs. adaptive queries. We show that $O(\log n)$ adaptive queries to a StoqMA or QMA oracle is equivalent to $\text{poly}(n)$ parallel queries to the oracle. Formally:

\[\text{pStoqMA[log]} = P^{\text{StoqMA}} \text{ and } P^{\text{QMA}[\text{log}]} = P^{\text{QMA}}.\]

**Result 2:** Complexity of $\text{v-APX-SIM}$ under simulations. We show that the complexity of a seemingly easier problem, $\text{v-APX-SIM}$ (see proof techniques), is preserved under simulations (in the sense of [3]).

**Result 3:** Complexity of $\text{APX-SIM}$ for physical Hamiltonians. Leveraging Result 1, we show that $P^{\text{QMA}[\text{log}]}$-complete for spatially-sparse $H$.

Combining with Result 2, APX-SIM remains $P^{\text{QMA}[\text{log}]}$-complete for any Hamiltonian family which can efficiently simulate spatially-sparse $H$.

**Punchline:** APX-SIM is $P^{\text{QMA}[\text{log}]}$-complete on physically motivated models like the Heisenberg anti-ferromagnetic interaction on a 2D lattice.

Proof techniques

Our approach proceeds in two high-level steps:

1. **Give $P^{\text{QMA}[\text{log}]}$ an equivalent characterization in terms of polynomially many parallel queries, i.e. $P^{\text{QMA}}$, which eases the analysis of using Ambainis’s [1] query Hamiltonian construction (Result 1).**
2. **We wish to apply the “simulation” framework of [3] to show that APX-SIM is $P^{\text{QMA}}$-complete on physically motivated $H$. Four substeps:**
   a. Introduce intermediary problem, $\text{v-APX-SIM}$ (see def. below).
   b. Show that simulation preserves the complexity of $\text{v-APX-SIM}$ (Result 2).
   c. Show that $\text{v-APX-SIM}$ is $P^{\text{QMA}[\text{log}]}$-complete for spatially sparse $H$.
   d. Apply existing simulation results [3] to obtain Result 3, i.e. that APX-SIM is $P^{\text{QMA}[\text{log}]}$-complete for various physically motivated models.

**Notes:**
- **Simulation [3]:** $H_Q$ is a simulation of Hamiltonian $H$ if there exists an efficiently computable local isometry $V = \otimes_i V_i$ that maps eigenvectors and eigenvalues of $H$ to those of $H_Q$ with “sufficiently small errors”.
- **v-APX-SIM:** Defined as APX-SIM but with “$\nu$ low-energy states $|\psi\rangle$” in the YES case. This problem is more robust than APX-SIM to perturbations in the ground space.

Complexity of APX-SIM for physical $H$

**Theorem:** APX-SIM is $P^{\text{QMA}[\text{log}]}$-complete even for $H$ restricted to a spatially-sparse interaction graph (in the sense of [5]).

**Proof:** We modify the hardness construction from Result 1 so that it is spatially-sparse. There are 3 Hamiltonian terms $H_{\text{base}} = H_{\text{Cook-Levin}} + H_{\text{queries}} + H_{\text{data}}$:

- **$H_{\text{Cook-Levin}}$** is already spatially sparse (on a 2D lattice, in fact) – another benefit of our modifications in proving Result 1.
- Since $k$-LH is QMA-hard for $H$ on a 2D lattice [5], we assume all query instances $H_{\text{query}}$ are spatially-sparse.
- But, $H_{\text{queries}}$ also has an answer register such that answer qubit $X_i$ interacts with every qubit in $H_{\text{query}}$; this is not spatially-sparse. We “spread out” the answer register to a grid of qubits overlaid on the query register, and introduce a stabilizing term $H_{\text{data}}$.

Combined with Result 2 and prior results for spatially-sparse Hamiltonians, we obtain many corollaries. Here are a few:

**Corollaries:** APX-SIM is $P^{\text{QMA}[\text{log}]}$-complete even for $H$ of XY interactions; for $H$ of Heisenberg interactions; or for $H$ on a 2D square lattice.

Discussion

**Key takeaways:**
1. The natural problem of simulating $O(\nu)$-qubit measurements against ground states of physically motivated systems, such as the Heisenberg $XY$ and antiferromagnetic interaction on a 2D lattice, is harder than QMA.
2. Working with parallel queries, as opposed to adaptive queries, simplifies circuit-to-Hamiltonian constructions.

**Open questions:**
- What other results for $k$-LH / QMA can be extended to $P^{\text{QMA}[\text{log}]}$?
- Identify additional $P^{\text{QMA}[\text{log}]}$-complete problems and physical inputs.
- What happens if we switch the $P$ base with other classes (classical or quantum) (ex. $B^{\text{QMA}[\text{log}]}$)?
- What if we use other quantum oracles (ex. $P^{\text{QIP}[\text{log}]}$)?

References


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