Oracle complexity classes and local measurements on physical Hamiltonians

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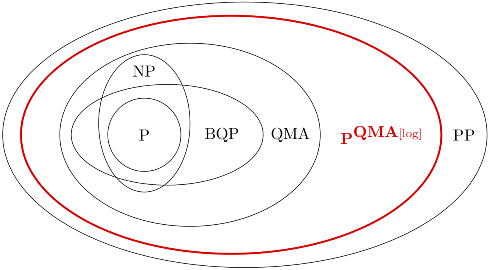
Two perspectives

Physical problem:

Approximate Simulation Problem (APX-SIM)

Estimate a measurement on the ground space of a local Hamiltonian.

Complexity Class:



Definitions

Approximate Simulation Problem (APX-SIM)

Estimate a measurement on the ground space of a local Hamiltonian.

Definition (**APX-SIM**$(H, A, k, l, a, b, \delta)$ [Ambainis, 2014])

Given:

- k-local Hamiltonian H on n qubits
- l-local observable A
- $a, b, \delta \in \mathbb{R}$ such that $b - a \geq \frac{1}{\text{poly}(n)}$ and $\delta \geq \frac{1}{\text{poly}(n)}$,

Decide:

- If H has a ground state $|\psi\rangle$ satisfying $\langle\psi| A |\psi\rangle \leq a$, output YES.
- If for all $|\psi\rangle$ satisfying $\langle\psi| H |\psi\rangle \leq \lambda(H) + \delta$, it holds that $\langle\psi| A |\psi\rangle \geq b$, output NO.

Definitions

Definition ($P^{QMA[\log]}$ [Ambainis, 2014])

$\mathbf{P^{QMA[log]}}$ is the class of decision problems decidable by a P machine with the ability to query a QMA oracle up to $O(\log n)$ times.

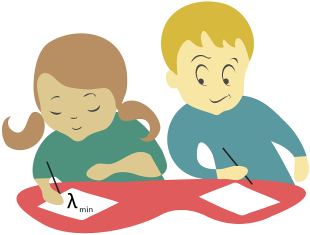
Intuitively,

$P^{QMA[\log]}$ is slightly harder than / above QMA.

Formally,

$QMA \subseteq P^{QMA[\log]} \subseteq PP$ [G., Y., 2016].

($QMA \subseteq PP$ previously known [Kitaev, Watrous, 2000])



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Analogue of classical $\mathrm{P}^{\mathrm{NP[log]}}$, from the study of "bounded query computations" in the 80s (nice survey by [Wagner, 1988]).

ex. Does the largest clique have odd size?

ex. Is the optimal MAX-SAT solution unique?

$\mathrm{P}^{\mathrm{QMA[log]}}$ is interesting because

- (like $\mathrm{P}^{\mathrm{NP[log]}}$) Complexity theory.
- (unlike $\mathrm{P}^{\mathrm{NP[log]}}$) It characterizes physically interesting problems: APX-SIM, estimating 2-point correlation functions, estimating spectral gaps, . . .

  [Ambainis, 2014], [G., Y. 2016]

History

k-LH: Given k-local Hamiltonian H, is $\lambda_{\text{min}} \leq a$ or $\lambda_{\text{min}} \geq b$, for $b - a \geq \frac{1}{\text{poly}(n)}$?

Estimating λ_{min} of local Hamiltonians is hard:

- k-LH is QMA-complete for $k \geq 5$ [Kitaev 99]
- for $k \geq 2$ [KKR06]

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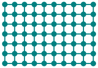
Spatially-local Hamiltonians are hard:

- for Hamiltonians restricted to nearest-neighbor interactions on a 2D grid of qubits [OT08]



- for nearest-neighbor interactions on a 1D line of qudits [AGIK09, Nag08, HNN13]

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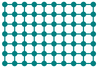

- for nearest-neighbor interactions on a 1D line of qudits [AGIK09, Nag08, HNN13]

Hamiltonians with restricted types of interactions are hard:

- the XY model plus 1-local terms [BL08]
- the Heisenberg model [SV09]

History

APX-SIM is $\mathsf{P}^{\mathsf{QMA[log]}}$-complete

- for $O(\log n)$-local Hamiltonians and observables [Ambainis, 2014]
- for 5-local Hamiltonians and single-qubit measurements! [G., Y., 2016]

. . . ?

Our results

Theorem (Adaptive vs. parallel queries)

$\mathsf{P}^{\mathsf{QMA}[\log]} = \mathsf{P}^{||\mathsf{QMA}}$ *and* $\mathsf{P}^{\mathsf{StoqMA}[\log]} = \mathsf{P}^{||\mathsf{StoqMA}}$

Theorem (Physical Hamiltonians)

$\mathrm{APX\text{-}SIM}$ *is* $\mathsf{P}^{\mathsf{QMA}[\log]}$*-complete for any family of Hamiltonians which can efficiently simulate spatially sparse Hamiltonians.*

This includes the 2D Heisenberg model [Cubitt, Montanaro, P., 2018].
Along the way: A full classification of $\mathrm{APX\text{-}SIM}$; show it is P, $\mathsf{P}^{\mathsf{NP}[\log]}$, $\mathsf{P}^{\mathsf{StoqMA}[\log]}$, or $\mathsf{P}^{\mathsf{QMA}[\log]}$-complete for successive Hamiltonian families.

Theorem (1D line)

$\mathrm{APX\text{-}SIM}$ *is* $\mathsf{P}^{\mathsf{QMA}[\log]}$*-complete for Hamiltonians on a 1D line of qudits and 1-local observables ($d = 8$).*

First result: Adaptive vs. parallel queries

Theorem

$P^{QMA[\log]} = P^{||QMA}$ *and* $P^{StoqMA[\log]} = P^{||StoqMA}$

Definition

$P^{||QMA}$ is the same as $P^{QMA[\log]}$ but with up to polynomially many parallel / non-adaptive queries.

Analogous classical result: $P^{NP[\log]} = P^{||NP}$ [Beigel, 1991].

Motivation: This will simplify our other proofs!

Forward direction, $P^{QMA[\log]} \subseteq P^{||QMA}$, is easy.
Proof exactly the same as for $P^{NP[\log]} \subseteq P^{||NP}$.

Ask all possible $2^{\log n} = \text{poly}(n)$ adaptive queries at once using the using the $P^{||QMA}$ machine.

Novel proof technique $\mathrm{P}^{\mathrm{QMA[log]}} \supseteq \mathrm{P}^{||\mathrm{QMA}}$

Classically, show $\mathrm{P}^{\mathrm{NP[log]}} \supseteq \mathrm{P}^{||\mathrm{NP}}$ by using NP oracle to binary search the number of parallel queries which are YES-instances.

This technique fails in the quantum case!
The P machine may make invalid queries, i.e which violate the promise of k-LH, with $\lambda_{\min} \in (a, b)$,
and the oracle is unpredictable/arbitrary given when invalid queries.
Important whenever the oracle corresponds to a promise problem.

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New technique (indirect):
We know $\mathrm{APX\text{-}SIM} \in \mathsf{P}^{\mathsf{QMA[log]}}$, so we prove $\mathrm{APX\text{-}SIM}$ is $\mathsf{P}^{||\mathsf{QMA}}$-hard, implying $\mathsf{P}^{\mathsf{QMA[log]}} \supseteq \mathsf{P}^{||\mathsf{QMA}}$.

APX-SIM is $\mathsf{P}^{||\mathsf{QMA}}$-hard (to show $\mathsf{P}^{\mathsf{QMA[log]}} \supseteq \mathsf{P}^{||\mathsf{QMA}}$)

Similar to proof that APX-SIM is $\mathsf{P}^{\mathsf{QMA[log]}}$-hard for $O(1)$-local input from [G., Y., 2016].

Use Kitaev Hamiltonian to correctly simulate P machine,

and use H_{query} of [Ambainis, 2014] to enforce correct query answers.

But, the Hamiltonian for parallel queries is much simpler than before:

$$H_{\text{query}} = \sum_{i=1}^{m} \frac{1}{4^{i-1}} \sum_{y_1,\ldots,y_{i-1}} \bigotimes_{j=1}^{i-1} |y_j\rangle\langle y_j|_{\mathcal{X}_j} \otimes \left(2\epsilon\,|0\rangle\langle 0|_{\mathcal{X}_i} \otimes I_{\mathcal{Y}_i} + |1\rangle\langle 1|_{\mathcal{X}_i} \otimes H_{\mathcal{Y}_i}^{i,y_1\cdots y_{i-1}}\right)$$

$$\longrightarrow H'_{\text{query}} = \sum_{i=1}^{m} (2\epsilon\,|0\rangle\langle 0|_{\mathcal{X}_i} \otimes I_{\mathcal{Y}_i} + |1\rangle\langle 1|_{\mathcal{X}_i} \otimes H^{i}_{\mathcal{Y}_i})$$

Other nice things:

- Simplifying this proof simplifies proofs of later results.
- Hardness holds for $b - a = \Omega(1)$ because we used Cook-Levin instead of Kiteav (it's a P machine!).

Second result: Physical Hamiltonians

Theorem

APX-SIM is $\mathsf{P}^{\mathsf{QMA[log]}}$-complete for any family of Hamiltonians which can efficiently simulate spatially sparse Hamiltonians.

Simulation: in the sense of [Cubitt, Montanaro, P., 2017].
Hamiltonian H' simulates H if there exists a local isometry mapping the low-energy space of H to the low-energy space of H' with sufficiently small errors.

ex. [Cubitt, Montanaro, P., 2017] show that Hamiltonians in the XY model can efficiently simulate spatially sparse Hamiltonians.

1. Show APX-SIM is closed under simulations.

This is non-obvious, because the definition of APX-SIM isn't robust to perturbations in the YES case.

Recall: APX-SIM

- If H has a ground state $|\psi\rangle$ satisfying $\langle\psi| A |\psi\rangle \leq a$, output YES.
- If for all $|\psi\rangle$ satisfying $\langle\psi| H |\psi\rangle \leq \lambda(H) + \delta$, it holds that $\langle\psi| A |\psi\rangle \geq b$, output NO.

We show a related, seemingly easier problem is also $\mathsf{P}^{||\mathsf{QMA}}$-complete:

Definition (∀-APX-SIM)

If for all $|\psi\rangle$ satisfying $\langle\psi| H |\psi\rangle \leq \lambda(H) + \delta$,

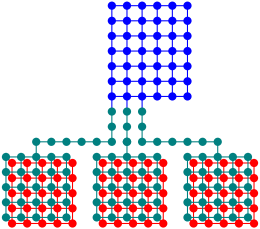
- $\langle\psi| A |\psi\rangle \leq a$, output YES.
- $\langle\psi| A |\psi\rangle \geq b$, output NO.

1. Show APX-SIM is closed under simulations.
2. Show APX-SIM is $\mathsf{P}^{||\mathsf{QMA}}$-hard for spatially-sparse Hamiltonians.

We make our $\mathsf{P}^{||\mathsf{QMA}}$-hardness construction from Theorem 1 spatially-sparse by:

- $H_{\text{Cook-Levin}}$ is already on a 2D grid.
- Reduce the QMA oracle queries to 2D [Oliveira, Terhal 2008], which H'_{query} acts on, disjointly.
- Design a stabilizer term, H_{stab}, to join the two.

The parallel/non-adaptive queries are critical to keep structure simple.



$$H = H_{\text{Cook-Levin}} + H''_{\text{query}} + H_{\text{stab}}$$

1. Show APX-SIM is closed under simulations.
2. Show APX-SIM is $\mathsf{P}^{||\mathsf{QMA}}$-hard for spatially-sparse Hamiltonians.
3. Use complexity classifications from [Cubitt, Montanaro, P., 2017].

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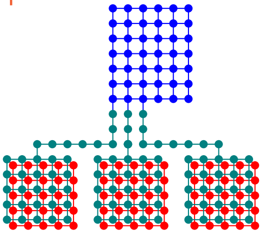


$$H = H_{\text{Cook-Levin}} + H''_{\text{query}} + H_{\text{stab}}$$

Third result: 1D line

Theorem

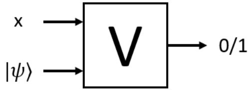
APX-SIM is $\mathsf{P}^{\mathsf{QMA[log]}}$-complete for Hamiltonians on a 1D line of qudits and 1-local observables ($d = 8$).

For previous results, we relied on H_{Kitaev} or $H_{\text{Cook-Levin}}$, and on the query Hamiltonian construction of [Ambainis, 2014].

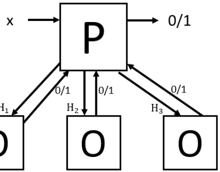
Now, we use the 1D circuit-to-Hamiltonian construction for QMA-hardness from [Hallgren, Nagaj, Narayanaswami, 2013]:

$$\cdots \otimes \otimes \| ▶ ◎ | □ ◎ | □ ○ \| ○ ○ | ○ ○ | ○ ○ \| ○ ○ \cdots ,$$

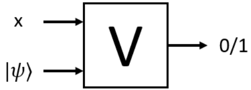
But, the query Hamiltonian isn't 1D, so need a new solution.



A QMA verifier, accepting input x and proof $|\psi\rangle$.



Previous constructions: A $\mathrm{P}^{||\mathrm{QMA}}$ circuit, asking queries H_i to the oracle O, recieving answer bits.



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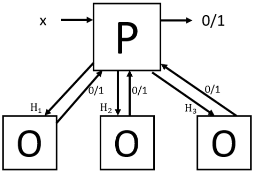
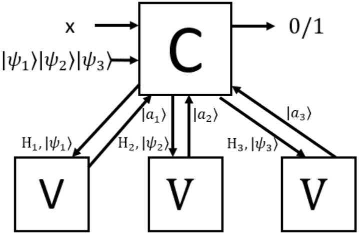


Previous constructions: A $P^{||QMA}$ circuit, asking queries H_i to the oracle O, recieving answer bits.



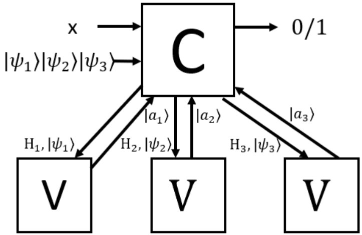
We treat the $P^{||QMA}$ circuit like a "big" quantum verification circuit, accepting input x and proofs $|\psi_i\rangle$.

Instead of queries to an oracle, it runs QMA verifier V as a subroutine, returning answers in superposition.

So, C is *like* a QMA verifier with input x and proof $|\psi_1\psi_2\ldots\rangle$.

And we apply the 1D circuit-to-Hamiltonian construction of [HNN13].

Replacing the query Hamiltonian



We treat the $\mathsf{P}^{||\mathsf{QMA}}$ circuit like a "big" quantum verification circuit, accepting input x and proofs $|\psi_i\rangle$.

Instead of queries to an oracle, it runs QMA verifier V as a subroutine, returning answers in superposition.

We still need to force the prover to be honest — previously enforced by the query Hamiltonian.
Even if query i is a YES-instance, the prover may send a failing proof $|\psi_i\rangle$.

So, we add "sifter" terms to each answer qubit, penalizing failing proofs.

$$H_{\text{out},i} = \epsilon \, |\blacktriangleright_0\rangle\langle\blacktriangleright_0|_{a_i}$$

Conclusion

Takeaways:

- $\mathrm{APX\text{-}SIM}$ is arguably more physically motivated than k-LH, and it's $\mathsf{P}^{\mathsf{QMA[log]}}$-complete.
- It's even $\mathsf{P}^{\mathsf{QMA[log]}}$-complete when estimating $O(1)$-qubit measurements against ground states of systems as simple as the 2D XY model or the 1D line.
- $\mathsf{P}^{\mathsf{QMA[log]}} = \mathsf{P}^{||\mathsf{QMA}}$. Use whichever definition is most convenient.

Open questions:

- Identify additional physical problems and inputs characterized by $\mathsf{P}^{\mathsf{QMA[log]}}$ and similar classes.
- Further study classes beyond $\mathsf{QCMA}, \mathsf{QMA}, \mathsf{QMA(2)}, \ldots$.
  ex. $\mathsf{P}^{\mathsf{QMA[log]}}, \mathsf{P}^{\mathsf{QMA(2)[log]}}, \ldots$
  ex. $\mathsf{QCPH}, \mathsf{QPH}$, a quantum bounded query hierarchy, ...
  And what can they tell us about more fundamental questions?